Engineering Maths First Aid Kit

8.3

Linearity rules

Introduction

There are two rules known as **linearity rules** which, when used with a Table of Derivatives, enable us to differentiate a wider range of functions. These rules are summarised here.

1. Some notation

Before we look at the rules, we need to be clear about the meaning of the notation $\frac{\mathrm{d}}{\mathrm{d}x}$.

When we are given a function $y(x)$ and are asked to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ we are being instructed to carry out an operation on the function $y(x)$. The operation is that of differentiation. A notation for this operation is used widely:

$\frac{\mathrm{d}}{\mathrm{d}x}$ stands for the operation: 'differentiate with respect to x'

For example, $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$.

2. Differentiation of a function multiplied by a constant

If k is a constant and f is a function of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf) = k\frac{\mathrm{d}f}{\mathrm{d}x}$$

This means that a constant factor can be brought outside the differentiation operation.

Example

Given that $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x^3) = 7 \times \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 7 \times 3x^2 = 21x^2$$

Given that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(8\sin x) = 8 \times \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 8\cos x$$

3. Differentiation of the sum or difference of two functions

If f and g are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(f-g) = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}g}{\mathrm{d}x}$$

This means that to differentiate a sum of two functions, simply differentiate each separately and then add the results. Similarly, to differentiate the difference of two functions, differentiate each separately and then find the difference of the results.

Example
Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = x^2 + x$.

Solution
We require $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x)$. The sum rule tells us to differentiate each term separately. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(x) = 2x + 1$$

So $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$.

Example
Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{2x} - \sin 3x$.

Solution
The difference rule tells us to differentiate each term separately.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} - \sin 3x) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) - \frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x) = 2\mathrm{e}^{2x} - 3\cos 3x$$

So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} - 3\cos 3x$$

Exercises
In each case use a Table of Derivatives and the rules on this leaflet to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = \mathrm{e}^{5x} + \cos 2x$
2. $y = x^2 - \sin x$
3. $y = 3x^2 + 7x + 2$
4. $y = 5$.
5. $y = 8\mathrm{e}^{-9x}$.

Answers
1. $5\mathrm{e}^{5x} - 2\sin 2x$, 2. $2x - \cos x$, 3. $6x + 7$, 4. 0, 5. $-72\mathrm{e}^{-9x}$.

